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RELATIONS AND FUNCTIONS
MODULE 2 (II)

REVISION

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How to prove a function, one to one and onto?

One-One Function (Injective mapping)

A function $f : A \rightarrow B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ and $f(x_1), f(x_2) \in B$.

$f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Onto Function (Surjective mapping)

If the function $f : A \rightarrow B$ is such that each element in B (co-domain) is the f image of at least one element in A, then we say that f is a function of A 'onto' B.

Thus, $f : A \rightarrow B$ is surjective iff $\forall\ b \in B$, $\exists$ some $a \in A$ such that $f(a) = b$.

A function $f : X \rightarrow Y$ is *one-one and onto* (or *bijective*), if f is both one-one and onto

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Q) **Are the following set of ordered pairs functions? If so, examine whether the mapping is injective or surjective.**

(i) $\{(x, y): x \text{ is a person, } y \text{ is the mother of } x\}$.

(ii) $\{(a, b): a \text{ is a person, } b \text{ is an ancestor of } a\}$.

Sol. (i) Given set of ordered pair is $\{(x, y): x \text{ is a person, } y \text{ is the mother of } x\}$

It represents a function. Here, the image of distinct elements of x under f are not distinct, so it is not injective but it is surjective.

(ii) Set of ordered pairs = $\{(a, b): a \text{ is a person, } b \text{ is an ancestor of } a\}$.

Here, each element of domain does not have a unique image. So, it does not represent

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Example 8

Show that the function $f : \mathbf{N} \to \mathbf{N}$, given by $f(x) = 2x$, is one-one but not onto.

Given,

$f(x) = 2x$

<u>One-One</u>

$f(x_1) = 2x_1$

$f(x_2) = 2x_2$

Putting

$f(x_1) = f(x_2)$

$2x_1 = 2x_2$

$x_1 = x_2$.

Hence, if $f(x_1) = f(x_2)$, $x_1 = x_2$

function f is **one-one**

<u>Rough</u>

One-one Steps:

1. Calculate $f(x_1)$

2. Calculate $f(x_2)$

3. Putting $f(x_1) = f(x_2)$

we have to prove $x_1 = x_2$

Further, f is not onto,

as for $1 \in \mathbf{N}$, there does not exist any x in $\mathbf{N}$

such that $f(x) = 2x = 1$.

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$f(x) = \frac{x-2}{x-3} : R - \{3\} \rightarrow R - \{1\}$. Prove f is one to one and onto

Let $f(x_1) = f(x_2), x_1, x_2 \epsilon R - \{3\}$

$$\frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

$(x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3)$

$x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 3x_2 - 2x_1 + 6$

$-x_1 = -x_2$

$x_1 = x_2$

Hence f is one to one

Let $y = \frac{x-2}{x-3}$

$x = \frac{3y-2}{y-1}, \quad y \epsilon R - \{1\}.$

For all elements

$y \epsilon$ codomain,

$x \epsilon$ *domain.*

Hence f is onto.

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Ex 1.2, 9

Let f: **N** → **N** be defined by $f(n) = \begin{cases} \frac{n+1}{2}, \text{ if } n \text{ is odd} \\ \frac{n}{2}, \text{ if } n \text{ is even} \end{cases}$ for all n ∈ **N**.

State whether the function f is bijective. Justify your answer.

Solution:

$$f(n) = \begin{cases} \frac{n+1}{2}, \text{ if } n \text{ is odd} \\ \frac{n}{2}, \text{ if } n \text{ is even} \end{cases} \quad \text{for all } n \in \mathbf{N}.$$

Check one-one

$f(1) = \frac{1+1}{2} = \frac{2}{2} = 1$ *(Since 1 is odd)*

$f(2) = \frac{2}{2} = 1$ *(Since 2 is even)*

Since, f(1) = f(2) but 1 ≠ 2

Both f(1) & f(2) have same image 1 ∴ f is **not one-one**

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Check onto

Let f(x) = y , such that y ∈ **N**

When n is odd	**When n is even**
$y = \frac{n+1}{2}$	$y = \frac{n}{2}$
$2y = n + 1$	$2y = n$
$2y - 1 = n$	$n = 2y$
$n = 2y - 1$	
Hence, for y is a natural number ,	Hence for y is a natural number
$n = 2y - 1$ is also a natural number	$n = 2y$ is also a natural number

Thus,

for every y ∈ **N**, there exists x ∈ **N** such that f(n) = y

Hence, f is **onto** . Hence the given function is not bijective.

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Example 12

Show that $f : \mathbf{N} \rightarrow \mathbf{N}$, given by $f(x) = \begin{cases} x+1, & if\ x\ is\ odd \\ x-1, & if\ x\ is\ even \end{cases}$ is both one-one and onto.

Check one-one

There can be 3 cases

1. x_1 & x_2 both are odd
2. x_1 & x_2 both are even
3. x_1 is odd & x_2 is even

> ***Rough***
>
> *One-one Steps:*
>
> *1. Calculate $f(x_1)$*
>
> *2. Calculate $f(x_2)$*
>
> *3. Putting $f(x_1) = f(x_2)$*
>
> *we have to prove $x_1 = x_2$*

If x_1 & x_2 are both odd

$f(x_1) = x_1 + 1$

$f(x_2) = x_2 + 1$

Putting $f(x_1) = f(x_2)$

$x_1 + 1 = x_2 + 1$

$x_1 = x_2$

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If x_1 & x_2 are both are even

$f(x_1) = x_1 - 1$

$f(x_2) = x_2 - 1$

If $f(x_1) = f(x_2)$

$x_1 - 1 = x_2 - 1$

$x_1 = x_2$

If x_1 is odd and x_2 is even

$f(x_1) = x_1 + 1$

$f(x_2) = x_2 - 1$

If $f(x_1) = f(x_2)$

$x_1 + 1 = x_2 - 1$

$x_2 - x_1 = 2$

which is impossible as difference between even and odd number can never be even

Hence, if $f(x_1) = f(x_2)$,

Then $x_1 = x_2$

∴ function f is **one-one**

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Check onto

$$f(x) = \begin{cases} x+1, & if\ x\ is\ odd \\ x-1, & if\ x\ is\ even \end{cases}$$

Let f(x) = y , such that y ∈ **N**

If x is odd

f(x) = x + 1

y = x + 1

y − 1 = x

x = y − 1

If x is odd, y is even

If x is even

f(x) = x − 1

y = x − 1

y + 1 = x

x = y + 1

If x is even, y is odd

$$x = \begin{cases} y-1, & if\ y\ is\ even \\ y+1, & if\ y\ is\ odd \end{cases}$$

Hence, if y is a natural number,

x will also be a natural number i.e. x ∈ **N**

Thus, f is **onto.**

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Example: Consider f: $N \rightarrow R$, $f(x) = 4x^2 + 12x + 15$
Show that f: $N \rightarrow$ R is one to one. Is f(x) onto? Why?

$$Let\ f(x_1) = f(x_2), x_1, x_2 \epsilon N$$
$$4{x_1}^2 + 12x_1 + 15 = 4{x_2}^2 + 12x_2 + 15$$

$$4({x_1}^2 - {x_2}^2) + 12(x_1 - x_2) = 0$$

$$4\,(x_1 - x_2)(x_1 + x_2 + 3) = 0$$

$$x_1 = x_2, \text{ OR } x_1 + x_2 = -3\ (rejected)$$

$$f\ is\ one\ to\ one.$$

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Example: Consider f: $N \to R$, $f(x) = 4x^2 + 12x + 15$
Show that f: $N \to$ R is one to one. Is f(x) onto? Why?

$$y = 4x^2 + 12x + 15$$

$$y = (2x+3)^2 - 9 + 15$$

$$x = \frac{\sqrt{y-6} - 3}{2},\ y \geq 6$$

but $y \in R \Rightarrow f$ *is not onto*

NOTE: If f : $N \to$ range of f, then f is onto.
Hence f is one to one and onto

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